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MATHEMATICAL MODEL FOR TIME DEPENDENT FLOWS IN A DENSELY PACKED POROUS LAYER

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ABSTRACT

This paper deals with the analytical investigation of a mathematical model to study the time dependent porous convection in a densely packed porous layer. Here, the medium is not only a two-phase system but also has a modulated environment. The model happens to be a Darcy model with the stress free planar boundaries. Stability analysis is performed in detail by the extended Stuart-Davis technique in order to know the qualitative as well as the quantitative features of the phenomenon. The time variation is introduced by oscillating the layer in the vertical direction. All the physical quantities like, volume expansion coefficient, kinematic viscosity, permeability, thermal diffusivity etc. are assumed to be constants. The profiles of velocity and temperature strongly depend on the type of modulation.

KEYWORDS: Time Dependent Flows, Densely Packed Porous Layer, Modulated Environment

1. INTRODUCTION

The classical problem of Rayleigh-B'enard convection in a horizontal and symmetric fluid/porous layer with temperatures prescribed on the boundaries has been the subject of numerous investigations owing to its wide applications in science, engineering and industrial areas. In fact, the study is of utmost importance in geophysical, astrophysical and heat transfer problems. Further, the extraction of energy from geothermal sources is the most promising one among the other methods. It is believed that the fluids in these reservoirs are highly permeable and consists of multicomponents rather than a single component. Actually, the study of convection in a fluid saturated porous layer is also of interest since it provides a convenient means for experimentally determining the nonlinear effects such as, the preferred cell pattern, heat and mass transport etc. The advantage of considering porous media is that the depth of the layer can be greatly increased (when compared to a fluid layer) since the frictional force is much larger. A thorough understanding of geophysical, astrophysical and meteorological convection process requires a good knowledge of the role played by gravity modulation on convective motions in fluid and porous layers under the influence of external constraints like rotation, salinity gradient, magnetic field etc. The present study is mainly concerned with time –dependent porous convection. The techniques and analysis with regard to stability or instability of a time-dependent basic state are fairly well understood. In such cases, the governing perturbation equations can be linearized and the onset conditions can be easily determined. This condition gives a sufficient condition for the basic state to be unstable. However, the case of instability problem with a time-dependent basic state is quite complicated. Therefore, the problem of instability of a time-dependent basic state has received less attention. The reason may be due to the difficulty encountered in the mathematical formulation of the problem with regard to the stability criterion, since the basic state can grow (or decay) simultaneously with the growth of a disturbance.

Gravity modulation occurs when a fluid / a porous layer is subject to vertical oscillations. In the case of a single component fluid under gravity modulation, there is an important distinction between modes of instability depending on whether the system is stable or unstable in the absence of modulation. The fundamental mode of instability exists if the unmodulated system is unstable. That is, this fundamental instability persists if the modulation amplitude is small enough. In general, a distribution of stratifying agencies that is convectively stable under constant gravity condition can be destabilized when a time-dependent component of the gravitational field is introduced. In fact, the effect of gravity modulation of a convectively stable configuration can significantly influence the stability of the system by increasing or decreasing its sucesseptibility to convection. It is interesting to note that this is a new way of controlling the stability of a system and is dependent on the magnitude of the amplitude or frequency of the modulation [1]. [2] have used the Galerkin's method to treat the linearized plane poiseuille flow with a modulated pressure gradient. [3] has made a study of the stability of a horizontal layer of fluid heated from below with a steady temperature difference between the walls of the layer and time-dependent sinusoidal perturbation applied to the wall temperatures. Encouraged by the crude experimental observations, [4] has made a study of the instability and growth of disturbances in a fluid layer in which the temperature gradient may be a function of depth and time. The foregoing discussion deals with linear theory only. [5] Has made a study of finite amplitude instability of time-dependent flows, by employing the extension of [6] technique. The amplitude equation is derived without making any assumption regarding the relative time scale of the basic state when compared to the growth rate of a disturbance. [7] Have made a theoretical as well as experimental observations regarding time-dependent heating experiments. [8] Have studied the effect of sinusoidal gravity modulation on the onset of solutal convection during vertical directional solidification of a binary alloy by neglecting the thermal effects. [9] Has investigated finite amplitude thermal convection with spatially modulated boundary temperatures. [10] have conducted experimental investigations and numerical simulations concurrently on two types of systems viz., (i) single and (ii) double-diffusive convection systems under gravity modulation, in order to study the resonance effect. [11] has made an extensive study of steady and oscillatory instability in a densely packed porous layer of infinite extent in the presence of rotation by considering linear as well as nonlinear system of equations A critical survey of the literature pertaining to the subject reveals that very sparse literature is available pertaining to the linear theory and no analytical work in the nonlinear regime is available, which is absolutely essential for the thorough understanding of the phenomenon. Therefore, the present investigation is carried out to provide an overall picture of the phenomenon by considering in detail the theory of time - dependent flows in a gravity modulated environment.

2. MATHEMATICAL FORMULATION

The physical configuration consists of a densely packed fluid saturated porous layer confined betweentwo infinite horizontal planes that are stress free and kept at constant temperatures. The mean distance between the two plates is'd'. Further, the lower plate is at a temperature T_h, whereas the upper plate is at a temperature T_c with $T_h>T_c$. The whole system is under gravity modulation so that

$$\vec{g} = g_0\left(1-g(t)\right)\hat{k} \qquad (1)$$

The governing equations of motion in the dimensionless form are:

The conservation of momentum

$$\left(-p^{-1}\frac{\partial}{\partial t}-\frac{1}{p_L}\right)\vec{q}+(1-g(t))R\theta\,\hat{k}-\nabla p=0 \qquad (2)$$

The conservation of energy

$$\left(-\frac{\partial}{\partial t}+\nabla^2\right)\theta-w\vec{T}_Z=\vec{q}\cdot\nabla\theta-\left(\overline{w\theta}\right)_Z \qquad (3)$$

The conservation of mass $\quad u_x+v_y+w_Z=0 \qquad (4)$

The equation of state $\quad \rho=\rho_0\left(1-\alpha\left(T-T_0\right)\right) \qquad (5)$

Here $\vec{g}$: the gravitational acceleration; g_0 : the constant part of the gravity; g(t): the time-dependent part due to oscillation; t:the time; $\hat{k}$: the unit vector in the z-direction; T: the temperature ; ρ, ρ_0 : the density and the mean density of the fluid; κ: the thermal diffusivity

Where (5) is in the dimensional form and α, ν, κ and ρ_0 are the constant coefficients of thermal expansion, kinematic viscosity, thermal diffusivity and reference density of the fluid respectively.

The scales used in making the equations dimensionless are:

Length : $\dfrac{d}{\pi}$; Velocity: $\dfrac{\kappa\pi}{d}$; Temperature: $\dfrac{\beta\, d}{\pi}$; Time: $\dfrac{d^2}{\pi^2\kappa}$; Pressure: $\dfrac{\rho_o\kappa\nu\pi^2}{d^2}$ (6)

In the process of non-dimensionalization the following dimensionless parameters appear:

$R=\dfrac{\alpha\beta g_0 d^4}{\pi^4\nu\kappa}$ Rayleigh number ;

$P = \dfrac{\nu}{\kappa}$ Prandtl number;

$P_L=\dfrac{\pi^2 k}{d^2}$: Porous Parameter / Darcy number; (7)

Further, a bar indicates the horizontal average. We write from the equation of the conservation of energy:

$$\left(-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}\right)\overline{T}=\left(\overline{w\theta}\right)_z \qquad (8)$$

Here $\overline{T}$ is the horizontal average of T and $\theta=T-\overline{T}$.

The analysis is restricted to the case of two-dimensional flow only under the limit of Prandtl number tending to infinity. The above assumption enables us to have the solutions in the closed form. We now write (2) in the component form so that

$$\left(-p^{-1}\frac{\partial}{\partial t}-\frac{1}{p_L}\right)u-\frac{\partial p}{\partial x}=0 \tag{9}$$

$$\left(-p^{-1}\frac{\partial}{\partial t}-\frac{1}{p_L}\right)w+(1-g(t))R\theta-\frac{\partial p}{\partial z}=0 \tag{10}$$

The pressure is eliminated, so that we have the following:

$$\frac{1}{P_L}\nabla^2 w-(1-g(t))R\theta_{xx}=0 \tag{11}$$

$$\left(-\frac{\partial}{\partial t}+\nabla^2\right)\theta-w\overline{T}_z=u\theta_x+w\theta_z-\left(\overline{w\theta}\right)_Z \tag{12}$$

$$\left(-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}\right)\overline{T}=\left(\overline{w\theta}\right)_Z \tag{13}$$

$$u_x+w_z=0 \tag{14}$$

The above set of equations will be solved in the domain- $\infty < x < \infty$, $0 \le z \le \pi$, $t \ge 0$.

Boundary Conditions

The horizontal boundaries are assumed to be planar, stress-free and perfectly conducting so that

$$w = u_z = \theta = 0 \quad \text{on } z = 0, \pi \tag{15}$$

$$\overline{T}(0,t)=\pi\,;\,\overline{T}(\pi,t)=0 \tag{16}$$

The problem will be completely specified only when the initial conditions are imposed. The two time regions emerge from the asymptotic analysis in the small parameter **(**R-R$_L$**)** ^{½} range where R_L is the critical value of the Rayleigh number corresponding to the linear theory. Since, the analysis is restricted to a single wave-number in the x-direction (i.e. only a single disturbance), the linear theory grows in the 'inner' layer. Thus the evolution of this mode will be considered in the outer region. Therefore, the appropriate initial conditions are given by

$$\left.\begin{aligned} w(x,z,o) &= W_0 \cos\alpha x \sin z,\\ \theta(x,z,0) &= \Theta_0 \cos\alpha x \sin z,\\ u(x,z,0) &= U_0 \sin\alpha x \cos z \end{aligned}\right\} \tag{17}$$

Where

$$\Theta_0 = \frac{\left(\alpha^2 + 1\right) W_0}{p_L \left(1 - g\left(0\right)\right) \alpha^2 R} \tag{18}$$

And $U0 = -W_0 \alpha^{-1}$ (19)

Are the compatibility conditions. On obtaining (18) and (19), the initial conditions (with finite number of modes) appropriate to the 'outer layer' are imposed and the results of the linear theory are anticipated.

3. THE EXPANSION PROCEDURE

In this section, the linearized problem is considered and the basic state solution is given by

$$\vec{q} = 0, \qquad \overline{T} = T_0\left(z\right) \qquad \text{and} \qquad \theta = 0 \tag{20}$$

Where, $T_0\left(z\right) = -\left(z - \pi\right)$ (21)

We now consider the linearized system given by

$$\frac{1}{P_L}\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)\hat{w}_1 + \alpha^2 R\left(1 - g\left(t\right)\right)\hat{\theta}_1 = 0 \tag{22}$$

$$\left(-\frac{\partial}{\partial t} + \frac{\partial^2}{\partial z^2} - \alpha^2\right)\hat{\theta}_1 + \hat{w}_1 = 0 \tag{23}$$

With

$$\hat{w}_1 = \hat{\theta}_1 = 0 \text{ On } z = 0,\pi \tag{24}$$

In the above set of equations, the x-dependence of all the variables is considered in the form $e^{i\alpha x}$ and also $\overline{T_{0z}} = -1$. We seek the solution to the system of equations (22) to (24) in the form;

$$\left(\hat{w}_1, \hat{\theta}_1\right) = \left[F_1\left(t\right), G_1\left(t\right)\right] \ \exp(a_0 t)\text{Sin}z \tag{25}$$

With the condition that F_1 and G_1must satisfy:

$$-\frac{1}{P_L}\left(\alpha^2 + 1\right)F_1 + \alpha^2 R\left(1 - g\left(t\right)\right)G_1 = 0 \tag{26}$$

$$-\left[\frac{d}{dt} + \left(a_0 + \alpha^2 + 1\right)\right]G_1 + F_1 = 0 \tag{27}$$

The solutions of (26) and (27) are:

$$G_1(t) = G_1(0)\exp\left[-\frac{\alpha^2 R\, P_L}{(\alpha^2+1)}\int_0^t g(s)\,ds\right] \tag{28}$$

$$F_1(t) = \frac{\alpha^2 R(1-g(t))\, P_L}{(\alpha^2+1)}\, G_1(t) \tag{29}$$

Where $$a_0 = \frac{\alpha^2 (R - R_L) P_L}{(\alpha^2+1)} \tag{30}$$

$$R_L = \frac{(\alpha^2+1)^2}{\alpha^2 P_L} \tag{31}$$

A careful observation of the solutions reveals that,

- The solutions are composed of the products of two parts viz. **exp(a_0t)** and a time bounded factor whenever g is integrable on [**0, ∞]**
- The permeability has its strong influence on all the factors and
- The factor exp (a_0t) represents the exponential growth of the perturbations and the growth rate.

5. RESULTS AND DISCUSSION

The results of the present study are presented in Figure 1 to Figure 5

In Figure 1 and Figure 2, the graphs of velocity profiles **F_1 (t) *vs.* t** for $\frac{1}{P_L} = 10^3$**,**

g (t) = 0.2exp (-t) and 0.2sin (t) respectively are plotted for different values of R. The pattern strongly depends on the type of modulation considered. For g (t) = 0.2exp (-t), F1 (t) decreases in the range $0 < t \le 5$ and then remains constant for all values 0f R. Further F_1 (t) decreases with R (Figure 1). In Figure 2, the curve is sinusoidal and in both cases the region of validity of t is quite large.

In Figure 3 the graph of G_1 (t) *vs* R for t = 5, 15, 25; and $\frac{1}{P_L} = 10^3$ is plotted. Obviously, the curve is linear and G_1 (t) decreases with R.

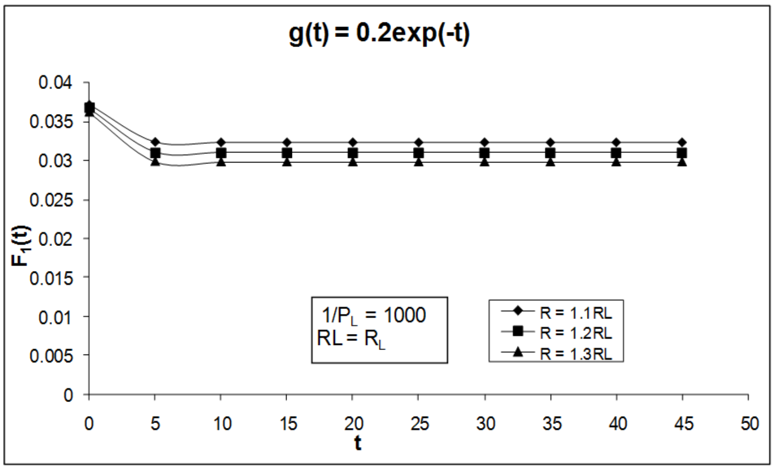

Figure 1

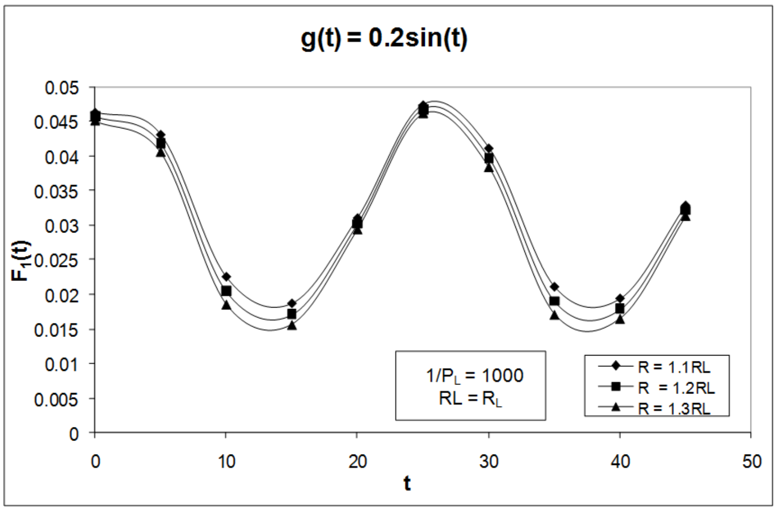

Figure 2

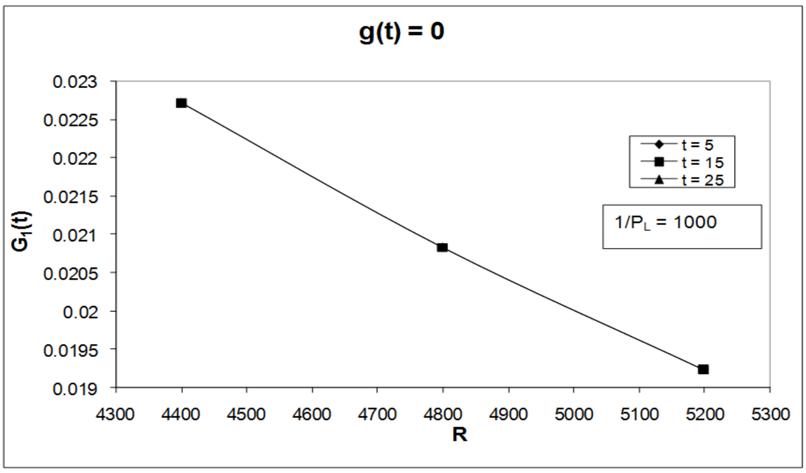

Figure 3

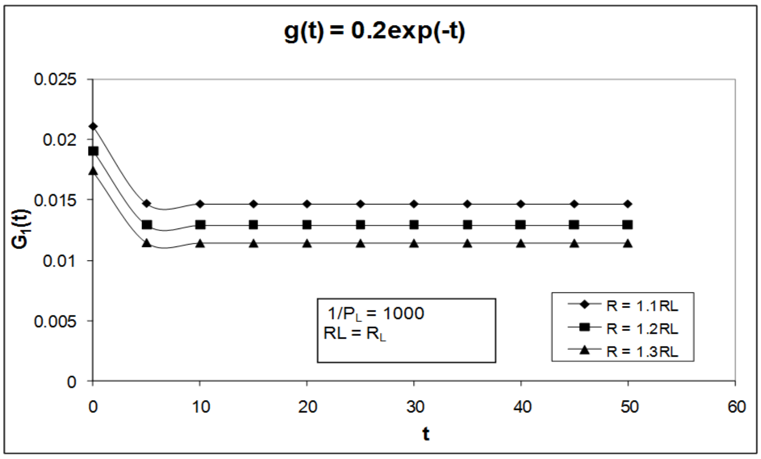

Figure 4

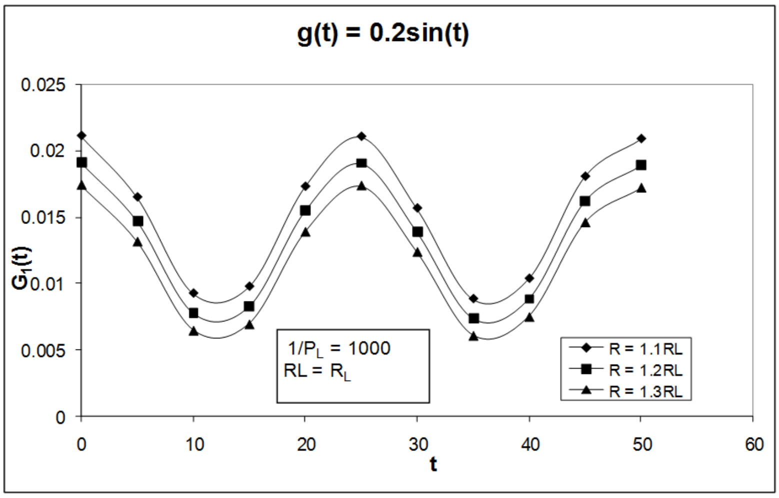

Figure 5

In Figure 4 and Figure 5, the graphs of temperature profiles $G_1(t)$ *vs* t of the modulation g(t) = 0.2exp(-t), 0.2sin(t) for $\frac{1}{P_L} = 10^3$ and different values of R is plotted**.** In Figure 4 G_1 (t) decreases with t initially and thereafter remains constant. But, in Figure 5 the profile is sinusoidal. The results are in agreement with the viscous case[5] i.e at $\frac{1}{P_L} = 0$ From the above results it is concluded that by a proper choice of g (t), P_L and R, it is possible to have a good control over the time-dependent porous convective phenomenon considered here.

The work throws light on the qualitative as well as the quantitative aspects of the problem.

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